Operator product expansion coefficients from the nonperturbative functional renormalization group

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Introduction: computing universal quantities with FRG

Systems near a quantum or classical continuous phase transition exhibit universality.

Functional renormalization group (FRG) has been able to reliably determine quantitatively universal quantities near strongly-coupled phase transitions:

- critical exponents; [e.g. among many: most recently De Polsi et al., PRE '20]
- amplitude ratios; [Berges et al., PR '02; Rançon et al., PRE '13; De Polsi et al., PRE '21]
- universal scaling functions. [e.g. Rose et al., PRB '15; Rose and Dupuis PRB '17]

Our work: go beyond and compute operator product expansion (OPE) coefficients in $O(N)$ theories.

Motivation: Operator Product Expansion

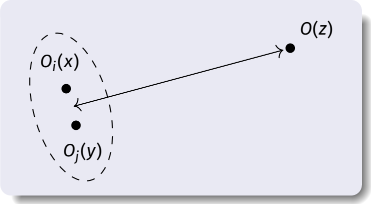
UV divergences → product of operators singular at short distance:

e.g. for a scalar field ϕ: $\lim_{y\to x}\langle\varphi(x)\varphi(y)\rangle = \infty \longrightarrow \varphi(x)\varphi(y)|_{y\to x} \neq \varphi(x)^2$.

Operator Product Expansion (OPE):

$$\text{for } y \to x, \quad O_i(x)O_j(y) = \sum_k \underbrace{f^{ijk}(x-y)}_{c\text{ number}}\overbrace{O_k(x)}^{\text{local operators}}$$

[Wilson '69, Kadanoff '70]



- sum over all local operators O_k;
- singularities included in f^{ijk}: Wilson coefficients;
- valid when inserted in correlation functions.

Verified to all orders in perturbation theory and in conformal field theories.

Applications of OPE

Renormalization theory [Brandt, Ann Phys '67], chromodynamics [Novikov et al., PR '78].

Ultracold gases: thermodynamic relations for $3d$ interacting fermions.

[Braaten and Platter, PRL '08]

$$\text{OPE: } \psi^\dagger_\sigma(\mathbf{R} - \tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) = \textstyle\sum_i C_i(\mathbf{r}) O_i(\mathbf{R}).$$

Operator identity: $C_i(\mathbf{r})$ determined by evaluating with few-body scattering states.

Result:

$$\psi^\dagger_\sigma(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r}) = \psi^\dagger_\sigma\psi_\sigma(\mathbf{R})$$

$$+ \mathbf{r}\cdot[\psi^\dagger_\sigma \overleftrightarrow{\boldsymbol{\nabla}}\psi_\sigma](\mathbf{R}) - \frac{r}{8\pi} g^2 \psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\uparrow\psi_\downarrow(\mathbf{R}) + O(r^2).$$

g: contact interaction.

E.g.: high-frequency tail of momentum distribution, $\rho_\sigma(\mathbf{k}) \sim C/k^4$.

$C = \int_{\mathbf{R}} \langle g^2 \psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\uparrow\psi_\downarrow\rangle$: Tan contact, $\partial_a \langle \hat{H} \rangle = (\hbar^2/4\pi m a^2) C$. [Tan, Ann Phys '08]

OPE in conformal field theories

In conformal field theories (CFT), OPE is the basis for conformal bootstrap (CB).

[Poland et al., RMP '19]

CFT: invariant under transforms that preserve angles. Then:

$$\langle O_i(x)O_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}},$$

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \frac{c_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}x_{23}^{\Delta_j+\Delta_k-\Delta_i}x_{13}^{\Delta_i+\Delta_k-\Delta_j}}.$$

- Δ_i: scaling dimension.
- $x_{12} = |x_1 - x_2|$.
- c_{ijk}: OPE coefficient!

OPE in CFTs

$$O_i(x)O_j(y) = \sum_k \frac{c_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} O_k(x) + \text{spinful fields.}$$

NB: true even for $d > 2$.

[Di Francesco, Mathieu, Sénéchal, CFT, Springer]

The O(N) model

Similar to φ^4 theory. $\boldsymbol{\varphi}$: N-component real field.

$$S[\boldsymbol{\varphi}] = \int \mathrm{d}^d x \left\{ \frac{1}{2} (\partial_\mu \boldsymbol{\varphi})^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

Phase transition controlled by the Wilson-Fisher fixed point for $d < 4$.

- $N = 1, 2, 3$: universality classes of physical systems (Ising, XY, Heisenberg).
- $N = \infty$: exact results.

At the phase transition: emergent conformal invariance!

Most relevant operators: $O_1 \propto \varphi_i$, $O_2 \propto \boldsymbol{\varphi}^2$. $\Delta_1 = (d - 2 + \eta)/2$, $\Delta_2 = d - 1/\nu$.

Question

Coefficient $c_{112} = ?$

- $4 - \epsilon$ expansion; [Dey et al., JHEP '17; Carmi et al., SciPost '21]
- Monte-Carlo; [Caselle et al. PRD '15; Hasenbusch, PRB '20]
- Conformal Bootstrap; [Kos et al. JHEP '16; Cappeli et al., JHEP '19]
- FRG. (Us !)

c_{112} coefficient with FRG

c_{112} can be deduced from correlation functions: [Pagani and Sonoda, PRD '20]

$$\text{for } |p_1| \gg |p_2|, \langle O_1(p_1) O_1(p_2) O_2(-p_1 - p_2)\rangle = \frac{c_{112}\times\text{const.}}{|p_1|^{d-\Delta_2}|p_2|^{d-2\Delta_1}}$$

Strategy: composite operators → add source h. [Rose, Léonard and Dupuis, PRB '15]

$$\mathcal{Z}[J,h] = \int \mathcal{D}[\varphi]\, e^{-S[\varphi]+\int_x (J\varphi + h\varphi^2)} \to \text{Legendre transf.: } \Gamma[\phi, h].$$

$$\langle \varphi_i(p_1)\varphi_i(p_2)\varphi^2(-p_1-p_2)\rangle = -\overbrace{G(p_1)}^{G:\ \text{propagator}}\,\underbrace{\Gamma^{(2,1)}_{ii}(p_1,p_2)}_{\Gamma^{(2,1)}_{ii} = \delta^3\Gamma/\delta\phi_i\delta\phi_i\delta h|_{\boldsymbol{\phi}=\text{const},h=0}:\ \text{vertex}}\, G(p_2)$$

Setting $p_2 = 0$, $p_1 = p \to 0$:

$$\boxed{c_{112} = \text{const.} \times \lim_{p\to 0} \frac{\Gamma^{(2,1)}_{ii}(p,0)}{|p|^{\Delta_2 - 2\Delta_1}}}$$

Momentum dependence → BMW scheme [Blaizot, Méndez-Galain and Wschebor, PLB '06]

- First determine $G(p)$ and $\chi_s = \langle \boldsymbol{\varphi}^2 \boldsymbol{\varphi}^2\rangle$ → normalization of operators, Δ_i.
- Then $\Gamma^{(2,1)}_{ii}(p,0) = \partial_{\phi_i}\Gamma^{(1,1)}_i(p) \to c_{112}$.

Results: c_{112} in the Ising model vs. d

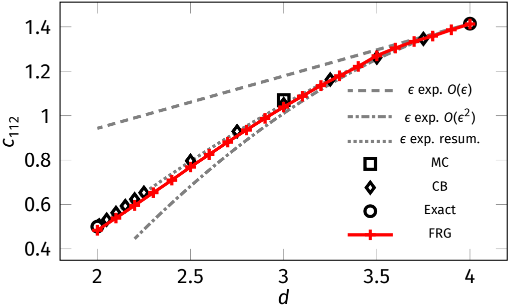
Ising model: universality class of $N = 1$.

Known values:

- $d = 2$, exact solution;
- $d = 4$, mean-field.

$2 < d < 4$:

- Monte-Carlo, Conformal bootstrap: numerically exact, but expensive;
- $\epsilon = 4 - d$ expansion: requires resummation and $d = 2$ result.

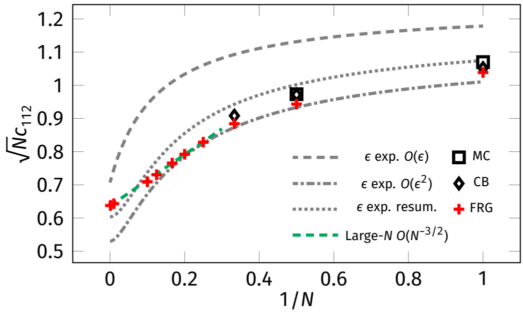


Results: c_{112} in the $3d$ O(N) model vs. N

Large N: $$c_{112} = \frac{2}{\pi}\frac{1}{\sqrt{N}} + \frac{24}{\pi^3}\frac{1}{N^{3/2}} + O\left(\frac{1}{N^{5/2}}\right)$$

[Lang and Rühl, Nucl. Phys. B '92]

→ rescaling: $\sqrt{N}c_{112}$.



- $N = 1, 2, 3$: agreement with CB and MC.
- $N \gtrsim 10$: agreement with large N results to next-to-leading order.
- Failure of ϵ expansion.

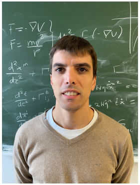
Conclusion

- FRG scheme to compute OPE coefficients via 3-point vertices;
- application to $c_{112}(d, N)$ for O(N) model;
- excellent agreement with results when known;
- versatility: tuning d and N is easy;
- perspectives: more complicated theories?

Read more in Phys. Rev. D **105**, 065020 (2022)!

Collaborators:

Thanks for your attention!



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Principle of minimum sensitivity