Real-time dynamics with FRG:
overcoming the burden of analytic continuation

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Exact Renormalization Group 2018

July 11, 2018

Introduction: what are QPTs?

Classical (continuous) phase transitions are well understood. (Landau, Kadanoff, Wilson)

- Landau theory: order parameter, symmetry breaking.
- Mean-field (usually) incorrect.
- Universal physics! E.g.: equation of state (Widom): $H = M^{\delta} f(tM^{-1/\beta})$.

What about $T = 0$ continuous quantum phase transitions (QPTs)?
Ground state *qualitatively* changes; gap vanishes.

E.g.: Mott insulator-superfluid transition.
Bosons trapped in an optical lattice:

Insulating Superfluid

[Greiner et al., Nature '02]

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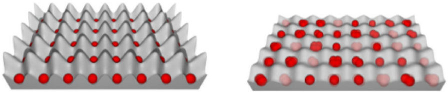

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[Greiner et al., Nature '02]

Outline of the presentation

Goal: understand universal properties of QPTs.

A lot is already known about the thermodynamics:
what about the dynamics?

Focus: the quantum $O(N)$ model in $2+1$ dimensions.

Outline:

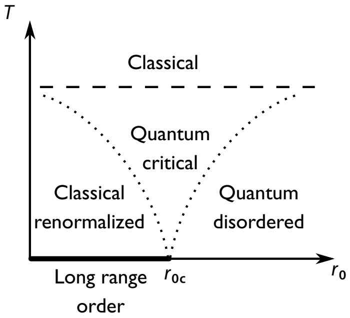
- presentation of the model, definition of quantities of interest;
- what are the issues posed by the dynamics;
- strategies to overcome the difficulty with FRG.

The O(N) model

Lorentz-invariant action; $\boldsymbol{\varphi}$: N-component real field ($\sim \varphi^4$ theory).

$$S[\boldsymbol{\varphi}] = \int_0^{\hbar\beta} d\tau \int d^2\mathbf{r} \left\{ \frac{1}{2} (\nabla \boldsymbol{\varphi})^2 + \frac{1}{2c_0^2} (\partial_\tau \boldsymbol{\varphi})^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

QPT in 2 space dimensions ≡ classical phase transition in 3 dimensions
→ quantum phase transition controlled by the 3D Wilson-Fisher fixed point.



Describes several phase transitions:

- $N = 2$: bosons in optical lattice; superconductor-insulator transition;
- $N = 3$: antiferromagnetic ordering in quantum magnets.

Dynamics

What about the dynamical properties of the system?

Information encoded in finite-momentum behavior of correlation functions!

- Excitation spectrum:
  - bound states; [**Rose**, Benitez, Léonard and Delamotte, PRD '16]
  - amplitude ("Higgs") mode. [**Rose**, Léonard and Dupuis, PRB '15]

- Transport properties, e.g. conductivity: $\Sigma(x, y)$: universal scaling function

$$\sigma(\omega, T) = \frac{e^2}{h}\Sigma\left(\frac{\omega}{\Delta}, \frac{k_B T}{\Delta}\right)$$

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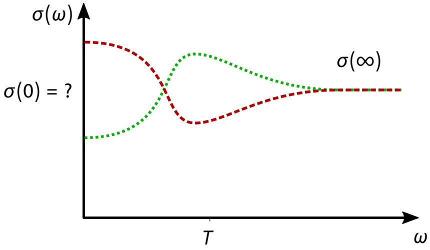
Conductivity

Graal: determine transport properties in the quantum critical regime for $\omega \lesssim T$.

2 possible scenarios:

[Damle and Sachdev, PRB '97]



Difficult: no quasiparticles, analytic continuation is hard.

Approaches:

- QMC (Sørensen, Chen, Prokof'ev, Pollet, Gazit, Podolsky, Auerbach);
- Holography (Myers, Sachdev, Witzack-Krempa);
- CFT (Poland, Sachdev, Simmons-Duffin, Witzack-Krempa);
- FRG (us!).

Difficulties in studying dynamics

A FRG approach to dynamics is hard for two reasons.

- We want the finite-momentum behavior of correlation functions
  → need to go beyond DE!
- The theory is formulated in imaginary time. Need to analytically continue...

  ...the results? (Successful at $T = 0$; unsatisfactory at $T > 0$.)

  ...the flow equations? (Extremely difficult!)

Our testbed: two-point correlation functions, conductivity.

Intermezzo: definition of conductivity?

Bosons ($N = 2$): current $\mathbf{j} \sim \mathrm{i}(\psi^* \nabla\psi - \psi\nabla\psi^*) \sim \varphi_i \varepsilon_{ij} \nabla \varphi_j$, $\begin{cases} \psi = \varphi_1 + \mathrm{i}\varphi_2, \\ \varepsilon_{ij} = -\varepsilon_{ji}. \end{cases}$

Generalization to $N > 2$:

$$j^a_\mu = \boldsymbol{\varphi} \cdot T^a \partial_\mu \boldsymbol{\varphi},$$

T^a: skew-symmetric matrix,

$\{T^a\}$: generators of SO(N) rotations.

→ $N(N-1)/2$ independent currents.

$[j^a_\mu = -\delta S/\delta A^a_\mu]$

Response to an external gauge field $A_\mu = A^a_\mu T^a$: given by conductivity tensor,

$$\langle j^a_\mu \rangle \sim \sigma^{ab}_{\mu\nu} \partial_t A^b_\nu.$$

Linear response theory

$$K^{ab}_{\mu\nu}(\mathbf{x} - \mathbf{x}') = \frac{\delta^{(2)} \ln \mathcal{Z}[\mathbf{A}]}{\delta A^a_\mu(\mathbf{x}) \delta A^b_\nu(\mathbf{x}')} \sim \langle j^a_\mu(\mathbf{x}) j^b_\nu(\mathbf{x}') \rangle$$

$$\sigma^{ab}_{\mu\nu}(\mathrm{i}\omega_n) = -\frac{1}{\omega_n} K^{ab}_{\mu\nu}(p_x = 0, p_y = 0, p_z = \omega_n)$$

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Effective action formalism

Conductivity → 4-point correlation functions $\langle j^a_\mu j^b_\nu \rangle$ (difficult!)

Trick: couple the action to two sources, gauge field **A** and linear source **J**:

$$\mathcal{Z}[\mathbf{J},\mathbf{A}] = \int \mathcal{D}[\mathbf{\Phi}] \exp\left(-S[\mathbf{\Phi},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\mathbf{\Phi}\right) \qquad [\partial_\mu \to D_\mu = \partial_\mu - A_\mu].$$

Effective action: Legendre transform of $\ln\mathcal{Z}$ wrt **J**, *but not* **A**:

$$\Gamma[\mathbf{\Phi},\mathbf{A}] = -\ln\mathcal{Z}[\mathbf{J},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\mathbf{\Phi}.$$

Conductivity ($\sim K^{ab}_{\mu\nu}$): now expressed with low-order vertices

$$K^{ab}_{\mu\nu} = -\Gamma^{(0,2)}_{a\mu,b\nu} + \Gamma^{(1,1)}_{i,a\mu}\left(\Gamma^{(2,0)}\right)^{-1}_{ij}\Gamma^{(1,1)}_{j,b\nu} \qquad \text{with} \qquad \Gamma^{(n,m)} = \left.\frac{\delta^{n+m}\Gamma}{\delta^n\Phi\,\delta^m A}\right|_{\mathbf{A}\to 0}.$$

Approximation scheme

Suitable FRG scheme:

- preserves gauge invariance ($\rightarrow$ excludes BMW!);
- has nontrivial momentum dependence.

LPA$''$: [Hasselmann, PRE '12].

Solution: NLPA / LPA$''$ *Ansatz*.

LPA$''$ for conductivity: [Rose and Dupuis, PRB '17].

$$\Gamma_k[\Phi] = \int_x \frac{1}{2}(\partial_\mu \Phi) \cdot Z_k(-\partial^2)(\partial_\mu \Phi) + \frac{1}{4}(\Phi \cdot \partial_\mu \Phi) Y_k(-\partial^2)(\Phi \cdot \partial_\mu \Phi) + U_k(\Phi^2)$$
$$+ \frac{1}{4} F^a_{\mu\nu} X_{1,k}(-D^2) F^a_{\mu\nu} + \frac{1}{4} F^a_{\mu\nu} T^a \Phi \cdot X_{2,k}(-D^2) F^b_{\mu\nu} T^b \Phi.$$

- $Z_k(\mathbf{p}^2)$, $Y_k(\mathbf{p}^2)$, $X_{1,k}(\mathbf{p}^2)$ and $X_{2,k}(\mathbf{p}^2)$ have non-trivial momentum dependence.
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$: building block for two $\mathcal{O}(A_\mu^2)$ terms.
- Gauge invariance preserved [Morris, N. Phys. B '00; Bartosch, PRB '13, ...].
- $\sigma(\omega)$ has a simple expression as a function of Z_k, $X_{1,k}$ and $X_{2,k}$.

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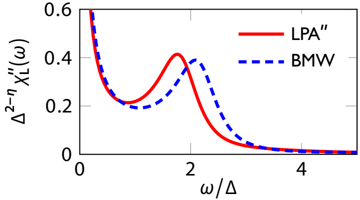
LPA″: benchmark

Two point correlation function: qualitative agreement with BMW.

Longitudinal susceptibility:
(N = 2, ordered phase)

[**Rose** and Dupuis, PRB '18]



Drawback: no field dependence.

- Disappointing value of η...

	LPA″	BMW	Bootstrap
ν	0.679	0.632	0.629971(4)
η	0.047	0.039	0.036298(2)

- ...and large-N only partially reproduced in symmetric phase.

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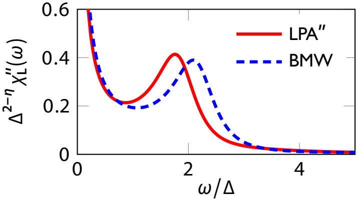

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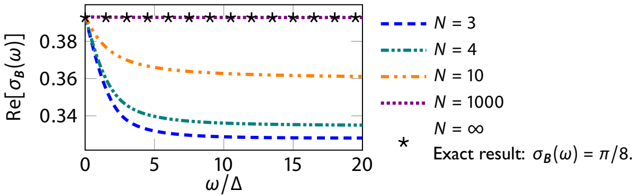
Results

Surprise in the ordered phase: (σ has then 2 components σ_A and σ_B)



$\sigma_B(\omega \to 0)$ does not numerically depend on N! [**Rose** et Dupuis, PRB '17]

> Conjecture: $\sigma_B(\omega \to 0) = \dfrac{\pi}{8}$ for all N: "superuniversality"!

Summary: "simple" scheme, gives access to $\omega > 0$... but *not* $T > 0$!

Non-local potentials have been considered in other contexts.

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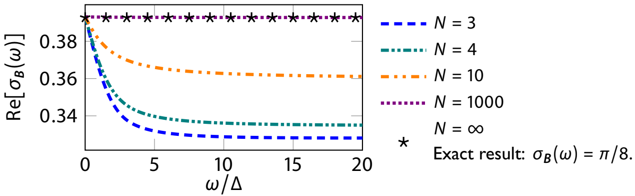

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Beyond LPA$''$: LPA$'$ continued?

Open problem: analytic continuation at $T > 0$ (Floerchinger, Pawlowski, Strodthoff)

$\rightarrow$ continue flow equations?

Issues:

- for $\mathrm{i}\omega_n \neq 2\pi \mathrm{i} nT$, $\sum_{\mathrm{i}\omega_m} \neq$ continuation: $\sum_{\mathrm{i}\omega_m}$ *must* be done analytically...
- ...but after continuation $\int_{\mathbf{q}}$ develop poles!

Solution: LPA$'$ "continued" (LPA$'$C)

$$\underbrace{\partial_k \Gamma_k}_{\text{full momentum dependence}} = \frac{1}{2} \operatorname{Tr} \partial_k R_k (\overbrace{\Gamma^{(2)}_{\mathrm{LPA}',k}}^{\mathrm{LPA}'\ \textit{Ansatz}} + R_k)^{-1}$$

RHS: LPA vertices and propagators; Θ regulator on $\mathbf{q}$ $\rightarrow$ Tr computed analytically

$$\partial_k \Gamma^{(2)}_k(\mathrm{i}\omega_n) = \partial_k F_k(\mathrm{i}\omega_n) \xrightarrow{\text{analytic continuation}} \partial_k \Gamma^{(2)}_k(z \in \mathbb{C}) = \partial_k F_k(z \in \mathbb{C})$$

F: explicit function of complex variable $z = \mathrm{i}\omega_n$.

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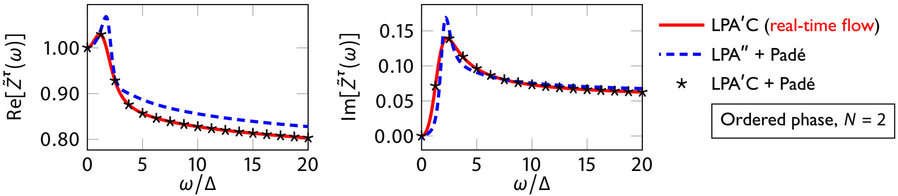
LPA$'$ continued: preliminary results at $T = 0$

Self-energy corrections $Z^\tau(\omega)$: $\Gamma^{(2)}(\mathbf{p} = 0, \omega) = -\omega^2 Z^\tau(\omega) + \Delta^2.$



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Lead for improvement: more involved *Ansatz* (DE?) in rhs

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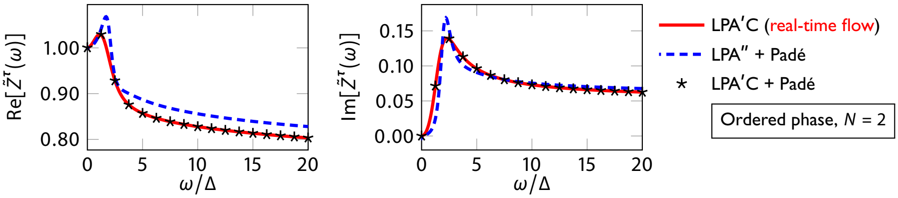

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Summary and conclusion

Takeaway messages:

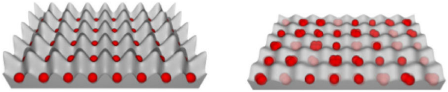
- dynamics: difficult to determine but rich in information;
- motivates development of new FRG schemes;
- FRG can deal with real-time flow equations!

Perspectives:

- improve LPA$'$C to explore finite-T physics;
- consider other transport coefficients, e.g. viscosity.

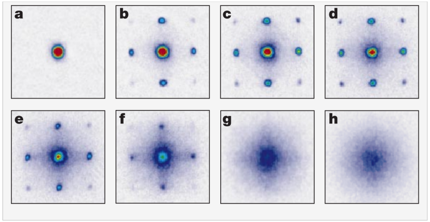
Experimental example: Mott insulator-superfluid transition

Bosons in an optical lattice:



Insulating Superfluid

Measuring the phase coherence of the condensate through interference:



From (a) to (h): potential depth increases.

[Greiner et al., Nature '02]

Conductivity: definition

O(N) symmetry → angular momentum conservation L, current: $\partial_t L + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$.

non-Abelian gauge field: $\partial_\mu \to D_\mu = \partial_\mu - A_\mu$.

$$A_\mu = A_\mu^a T^a \in \mathrm{so}(N) \qquad T^a\text{: } N(N-1)/2 \text{ generators, } T^a_{ij} = -T^a_{ji}$$

Current density $J^a_\mu = -\dfrac{\delta S}{\delta A^a_\mu} = j^a_\mu - A^a_\mu \boldsymbol{\varphi} \cdot T^a \boldsymbol{\varphi}$, $\qquad j^a_\mu = \boldsymbol{\varphi} \cdot T^a \partial_\mu \boldsymbol{\varphi}$

$N = 2$ (bosons): $\mathbf{j} \sim \mathrm{i}(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^*)$, $\quad \psi = \varphi_1 + \mathrm{i}\varphi_2$.

Linear response

$$K^{ab}_{\mu\nu}(\mathbf{x} - \mathbf{x}') = \langle j^a_\mu(\mathbf{x}) j^b_\nu(\mathbf{x}') \rangle - \delta_{\mu\nu} \delta(\mathbf{x} - \mathbf{x}') \langle T^a \boldsymbol{\Phi} \cdot T^b \boldsymbol{\Phi} \rangle$$

$$\sigma^{ab}_{\mu\nu}(\mathrm{i}\omega_n) = -\frac{1}{\omega_n} K^{ab}_{\mu\nu}(p_x = 0, p_y = 0, p_z = \mathrm{i}\omega_n)$$ tenseur de conductivité

Vertices' expression

Writing the vertices in the most general form, one has

$$\Gamma^{(2,0)}_{ij}(\mathbf{p},\mathbf{\Phi}) = \delta_{ij}\Gamma_A + \Phi_i\Phi_j\Gamma_B, \qquad \text{(inverse propagator!)}$$

$$\Gamma^{(1,1)}_{i,a\mu}(\mathbf{p},\mathbf{\Phi}) = \mathrm{i}p_\mu(T^a\mathbf{\Phi})_j\Psi_A,$$

$$\Gamma^{(0,2)}_{i,a\mu}(\mathbf{p},\mathbf{\Phi}) = \delta_{ab}[p_\mu p_\nu\Psi_B + \delta_{\mu\nu}\bar{\Psi}_B] + (T^a\mathbf{\Phi})\cdot(T^b\mathbf{\Phi})[p_\mu p_\nu\Psi_C + \delta_{\mu\nu}\bar{\Psi}_C],$$

where the Γs and the Ψs are functions of $\mathbf{p}^2$ and $\rho = \mathbf{\Phi}^2/2$.

Ward identities associated with gauge invariance indicates that only $\Gamma_{A,B}$ and $\Psi_{B,C}$ are independent.

Problem: regulator (~ mass) breaks down gauge invariance:

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{q}} \mathbf{\Phi}(\mathbf{q}) \cdot R_k(\mathbf{q}^2)\mathbf{\Phi}(-\mathbf{q}).$$

How to preserve gauge invariance?

Solution: make the regulator **A**-dependent!

[Morris, N. Phys. B '00] [Codello, Percacci et coll., EPJC '16] [Bartosh, PRB '13]

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{x}} \mathbf{\Phi}(\mathbf{x}) \cdot R_k(-\partial_\mu^2)\mathbf{\Phi}(\mathbf{x}) \to \Delta S_k[\mathbf{A}] = \frac{1}{2}\int_{\mathbf{x}} \mathbf{\Phi}(\mathbf{x}) \cdot R_k(-D_\mu^2)\mathbf{\Phi}(\mathbf{x})$$

Modified flow equations in presence of **A**.

RG approximation scheme

Which approximation procedure do we use?

First idea: BMW to obtain full momentum dependence (as done for the study of the Higgs mode).

Problem: it fails!

- Impossible to close the flow equations rigorously.
- Setting momenta to zero in flow equations breaks down gauge invariance.
- Vertices have a nontrival momenta dependence due to the derivative in j_μ^a...
- ...so it is not possible to close the equations without additional uncontrolled approximations...
- ...which break Ward identities!

RG approximation scheme

Which approximation procedure do we use?

First idea: BMW to obtain full momentum dependence (as done for the study of the Higgs mode).

Problem: it fails!

- Impossible to close the flow equations rigorously.
- Setting momenta to zero in flow equations breaks down gauge invariance.
- Vertices have a nontrival momenta dependence due to the derivative in j_μ^a...
- ...so it is not possible to close the equations without additional uncontrolled approximations...
- ...which break Ward identities!

LPA''

$$\Gamma_k[\Phi, \mathbf{A}] = \int_{\mathbf{x}} \frac{1}{2}(D_\mu \Phi) \cdot Z_k(-\mathbf{D}^2)(D_\mu \Phi) + \frac{1}{4}(\Phi \cdot \partial_\mu \Phi) Y_k(-\partial^2)(\Phi \cdot \partial_\mu \Phi) + U_k(\rho)$$
$$+ \frac{1}{4} F^a_{\mu\nu} X_{1,k}(-\mathbf{D}^2) F^a_{\mu\nu} + \frac{1}{4} F^a_{\mu\nu} T^a \Phi \cdot X_{2,k}(-\mathbf{D}^2) F^b_{\mu\nu} T^b \Phi.$$

Expression of conductivity within LPA''

$$\sigma_A(\omega) = 2\rho_0 Z(\omega^2)/(\omega + i0^+) + \omega[X_1(\omega^2) + 2\rho_0 X_2(\omega^2)],$$
$$\sigma_B(\omega) = \omega X_1(\omega^2).$$

$\rho = \Phi^2/2$, $\rho_{0,k}$: minimum of the potentiel.

Origins: Goldstone-modes controlled physics. Gauge-invariant action:

$$\Gamma^{\text{eff}}[\boldsymbol{\pi}, \mathbf{A}] = Z \int_{\mathbf{x}} [(\partial_\mu - A_\mu)\boldsymbol{\pi}]^2 + \cdots$$

Free bosons → σ_B computed via Wick's theorem,

$$\langle jj \rangle \sim \int_{\mathbf{q}} \Gamma^{(2,1)} G_{\text{T}}(\mathbf{q}) \Gamma^{(2,1)} G_{\text{T}}(\mathbf{p}+\mathbf{q}).$$

- Z factors disappear.
- $N = \infty$ result recovered.