Conductivity in the vicinity of a quantum critical point

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Outline of the talk

Several systems undergo a zero-temperature quantum phase transition when an external parameter is tuned.
Our goal: study universal transport properties of those transitions.

Here, we'll talk about the conductivity of the quantum O(N) model in $2+1$ dimensions.

Outline:

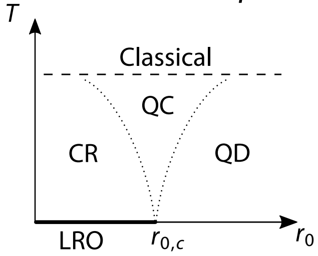
- we'll present the model, definitions and some general results;
- devise an ERG (or fRG, or NPRG...) scheme to compute conductivity;
- ...and show some results.

Introduction: the O(N) model

The action is

$$S[\boldsymbol{\varphi}] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{r}\, \frac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 + r_0 \boldsymbol{\varphi}^2 + u_0(\boldsymbol{\varphi}^2)^2.$$

$\boldsymbol{\varphi}$: N-component scalar field, β: inverse temperature, $d = 2$



Quantum phase transition controlled by the 3D Wilson-Fisher fixed point.

Describes SF-insulator transition for lattice bosons ($N = 2$), AF ordering for spins systems ($N = 3$).

Definition of the conductivity

Make the $O(N)$ symmetry local by adding a gauge field, $\partial_\mu \to D_\mu = \partial_\mu - A_\mu$.

$$A_\mu = A_\mu^a T^a \in \mathrm{so}(N) \qquad T^a\text{: } N(N-1)/2 \text{ generators, } T_{ij}^a = -T_{ji}^a$$

Current densities $J_\mu^a = -\dfrac{\delta S}{\delta A_\mu^a} = j_\mu^a - A_\mu^a \boldsymbol{\varphi}\cdot T^a \boldsymbol{\varphi}$, $\qquad j_\mu^a = \boldsymbol{\varphi}\cdot T^a \partial_\mu \boldsymbol{\varphi}$

$N = 2$ (bosons): $\mathbf{j} \sim \mathrm{i}(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*)$, $\quad \psi = \varphi_1 + \mathrm{i}\varphi_2$.

Linear response theory

$$K_{\mu\nu}^{ab}(\mathbf{x}-\mathbf{x}') = \left\langle j_\mu^a(\mathbf{x}) j_\nu^b(\mathbf{x}')\right\rangle - \delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{x}')\left\langle T^a\boldsymbol{\varphi}\cdot T^b\boldsymbol{\varphi}\right\rangle = \frac{\delta^{(2)} \ln Z}{\delta A_\mu^a(\mathbf{x})\delta A_\nu^b(\mathbf{x}')}$$

$$\sigma_{\mu\nu}^{ab}(\mathrm{i}\omega_n) = -\frac{1}{\omega_n} K_{\mu\nu}^{ab}(p_x = 0, p_y = 0, p_z = \mathrm{i}\omega_n) \qquad \text{conductivity tensor}$$

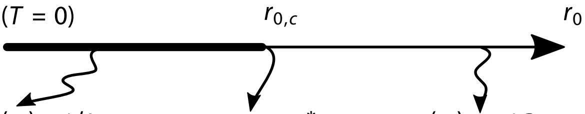
Conductivity: generalities

The conductivity tensor $\sigma^{ab}_{\mu\nu}$:

- is diagonal, $\sigma^{ab}_{\mu\nu} = \delta_{\mu\nu}\delta_{ab}\sigma^{aa}$;
- has two independent components, $\sigma^{aa}(\omega) = \begin{cases} \sigma_A(\omega) & \text{if } T^a\boldsymbol{\varphi} \neq 0, \\ \sigma_B(\omega) & \text{if } T^a\boldsymbol{\varphi} = 0; \end{cases}$
- in the disordered phase and at the QCP $\sigma_A = \sigma_B = \sigma$.

For $N = 2$, there is only one so(N) generator and the conductivity in the ordered phase reduces to σ_A.

Low frequency behavior: **[Gazit, Podolsky, Auerbach, PRB 13]**



$\sigma_A(\omega) \sim i/L\omega$ $\qquad$ $\sigma(\omega) = \sigma^*$ $\qquad$ $\sigma(\omega) \sim -iC\omega$

$\sigma_B(\omega) =$? (not much investigated as only exists for $N \geq 3$)

$\sigma^*/\sigma_q, C/L\sigma_q^2$ are universal! $\quad (\sigma_q = q^2/h)$ **[Fischer *et al.*, PRL 89]**

Goals

Objective: determine the universal scaling form of the conductivity.
Technically: compute 4 point correlation functions $\left\langle j_\mu^a j_\nu^b \right\rangle$.

Approaches:

- QMC **(Sørensen, Chen, Prokof'ev, Pollet, Gazit, Podolsky, Auerbach)**;
- AdS/CFT **(Myers, Sachdev, Witzack-Krempa)**;
- ERG (us!).

Effective action formalism

The partition function depends on two sources, the gauge field and **J** that couples linearly to $\boldsymbol{\varphi}$:

$$Z[\mathbf{J},\mathbf{A}] = \int D[\boldsymbol{\varphi}] \exp\left(-S[\boldsymbol{\varphi},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\boldsymbol{\varphi}\right).$$

The effective action is the Legendre transform of $\ln Z$ wrt **J** *but not* **A**:

$$\Gamma[\boldsymbol{\varphi},\mathbf{A}] = -\ln Z[\mathbf{J},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\boldsymbol{\varphi}.$$

$$K^{ab}_{\mu\nu} = \frac{\delta^{(2)}\ln Z}{\delta A^a_\mu \delta A^b_\nu} = -\Gamma^{(0,2)}_{a\mu,b\nu} + \Gamma^{(1,1)}_{i,a\mu}\left(\Gamma^{(2,0)}\right)^{-1}_{ij}\Gamma^{(1,1)}_{j,b\nu}$$

with

$$\Gamma^{(n,m)} = \frac{\delta^{(n+m)}\Gamma}{\delta^{(n)}\varphi\,\delta^{(m)}A}.$$

ERG formulation

ERG scheme: add a k-dependent infrared regulator to construct a family of theories that interpolate between MF ($k = \Lambda$) and exact solution ($k = 0$).

$$S \to S_k = S + \Delta S_k, \qquad \Gamma \to \Gamma_k, \qquad \Delta S_k = \frac{1}{2}\int_{\mathbf{q}} \boldsymbol{\varphi}(\mathbf{q}) \cdot R_k(\mathbf{q}^2)\boldsymbol{\varphi}(-\mathbf{q}),$$

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}[k\partial_k R_k \cdot (\Gamma_k^{(2,0)} + R_k)^{-1}]$$

Problem: how to preserve gauge invariance?

Make the regulator gauge dependent!

[Morris, N. Phys. B 00] [Codello, Percacci *et al.*, EPJC 16] [Bartosh, PRB 13]

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}) \cdot R_k(-\partial_\mu^2)\boldsymbol{\varphi}(\mathbf{x}) \to \Delta S_k[\mathbf{A}] = \frac{1}{2}\int_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}) \cdot R_k(-D_\mu^2)\boldsymbol{\varphi}(\mathbf{x})$$

Modified flow equations due to the presence of $\mathbf{A}$.

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RG approximation scheme

The flow for $\Gamma^{(n,m)}$ involves $\Gamma^{(n+1,m)}$ and $\Gamma^{(n+2,m)}$

$\Rightarrow$ how do we close the flow equations?

First idea: BMW to obtain full momentum dependence (as done for the study of the Higgs mode, **[FR, Léonard, Dupuis, PRB 15]**).

Problem: it fails!

- Impossible to close the flow equations rigorously.
- Setting momenta to zero in flow equations breaks down gauge invariance.

Derivative expansion scheme

We rather try a derivative expansion scheme and project the flow equation onto a gauge-invariant *Ansatz*. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ allows us to build two $O(A_\mu^2)$ gauge invariant terms:

$$\Gamma_k[\boldsymbol{\varphi}, \mathbf{A}] = \int_{\mathbf{x}} \frac{1}{2} Z_k(\rho)(D_\mu \boldsymbol{\varphi})^2 + \frac{1}{4} Y_k(\rho)(\partial_\mu \rho)^2 + U_k(\rho) \quad \text{(standard } O(\partial_\mu^2) \text{ DE)}$$
$$+ \frac{1}{4} X_{1,k}(\rho)\, \mathrm{Tr}(F_{\mu\nu}^2) + \frac{1}{4} X_{2,k}(\rho)(F_{\mu\nu}\boldsymbol{\varphi})^2.$$

$\rho = \boldsymbol{\varphi}^2/2$, $\rho_{0,k}$: minimum of the potential.

Expression of the conductivity within the DE

$$\sigma_A(\omega) = 2\rho_0 Z(\rho_0)/\omega + \omega[X_1(\rho_0) + 2\rho_0 X_2(\rho_0)],$$
$$\sigma_B(\omega) = \omega X_1(\rho_0).$$

Results

This simple DE scheme allows us to recover the low momenta physics! We retrieve the universal ratio C/L. Exact value for $N = \infty$, good agreement with MC ($\sim 5\%$) for $N = 2$.

N	2	3	4	1000	∞ (exact)
$C/NL\sigma_q^2 \quad (\sigma_q = q^2/h)$	0.105	0.0742	0.0598	0.0416	0.04167

The picture is more complicated in the critical regime.

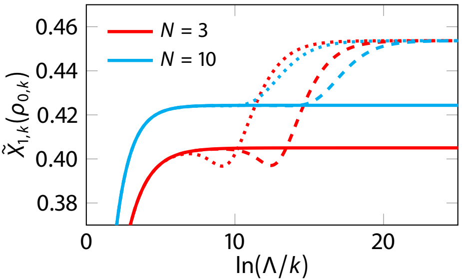
- DE is only valid at $\omega \ll k$.
- $\Gamma^{(0,2)}(\mathbf{p}) \sim 1/p$: divergence in the flow:

$$\sigma(\omega) \sim \tilde{X}^*_{1,\text{crit}} \frac{\omega}{k} \qquad \text{with } \tilde{X}^*_{1,\text{crit}} = \lim_{k\to 0} k X_1(\rho_{0,k}) \quad \text{(fixed point value).}$$

- Setting $\omega \sim k$ yields an estimate of the conductivity, $\sigma^* \sim \tilde{X}^*_{1,\text{crit}}$.

Similarly, in the ordered phase, $\sigma_B(\omega) \sim \tilde{X}^*_{1,\mathrm{ord}}\,\omega/k$.

$\tilde{X}^*_{1,\mathrm{ord}}$ is an universal number — verified for $N = \infty$, conjecture for $N < \infty$.



Full: QCP

Dashed: ordered

More surprising: $\tilde{X}^*_{1,\mathrm{ord}}$ numerically does not depend on N!

$$\sigma_B(\omega) = \frac{\pi}{8}\sigma_q \text{ for all } N\text{: “superuniversality”!}$$

Review and conclusion

- It is possible to devise a "gauge-invariant" ERG scheme to compute the conductivity.
- A simple derivative expansion allows to obtain results that compare well with MC.
- Results allow to make a conjecture on the universal behavior of σ_B...
- ...that needs to be confirmed with a momentum-dependent scheme we are currently developing.
- Long-term goal: $T > 0$!

A preprint will soon be available on the arXiv!