The nonperturbative functional renormalization group for classical and bosonic systems: overview and examples

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Introduction

Many physical systems can be described by field theories:

high-energy physics, quantum many-body systems, (classical) criticality...

Many theoretical (non-numerical) tools to tackle them:

e.g. perturbation theory, conformal field theory, renormalization group.

> Functional renormalization group (FRG):
> implementation of Wilson's renormalization group.

Different fields: different questions and problems!

Today: application of FRG to classical and boson-type problems.

Outline of the presentation

- Motivation: nonperturbative methods in statistical mechanics.
- Presentation of FRG.
- Derivative expansion: application to Ising model.
- Application 1: operator product expansion coefficients in the O(N) model.
- Application 2: Thermodynamics and dynamics near quantum criticality.

Reference reviews:

- Older, complete review: [Berges et al. PR '02].
- Useful for learning: [Delamotte `arXiv:cond-mat/0702365`].
- Recent review about applications [Dupuis et al. PR '21].

Strong correlations in statistical physics

When is mean-field (MF) valid, e.g. for classical Ising: $\boxed{H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_j \sigma_j}$?

Weak fluctuations:
can safely approximate $\sigma_i \simeq \langle \sigma \rangle$.

Mathematically:
$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-H[\{\sigma_i\}]} \simeq e^{-H[\{\sigma_i^*\}]}$
dominated by $\{\sigma_i^*\}$ that minimizes the free energy.

Low energy properties described by φ^4 theory: $\boxed{S[\varphi] = \int_x \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + r_0 \varphi^2 + u_0 \varphi^4}$.

- Mean field: $\mathcal{Z} \simeq e^{-S[\varphi^*]}$.
- Gaussian approx.: $\mathcal{Z} \simeq \int \mathcal{D}[\delta\varphi] \exp\left(- S[\varphi^*] - \int_{xy} \frac{1}{2} \delta\varphi_x \frac{\delta^2 S}{\delta\varphi_x \delta\varphi_y}\bigg|_{\varphi^*} \delta\varphi_y \right)$.

Breakdown of mean field theory

Mean field no longer valid when fluctuations away from $\langle\sigma\rangle$ dominate $\mathcal{Z}$.

When does this happen?

E.g. close to phase transition.

Fluctuations over all length scales $\lesssim$ correlation length ξ.

Ginzburg criterion (upper critical dimension)

$$\langle\sigma^2\rangle_c / \langle\sigma\rangle^2 \propto (T - T_c)^{(D-4)/2}.$$

- $D > 4$: fluctuations neglectable, MF valid at the transition.
- $D < 4$: fluctuations at all scales from a to ξ contribute to $\mathcal{Z}$: MF breaks down.

(a: lattice spacing, T_c: critical temperature.)

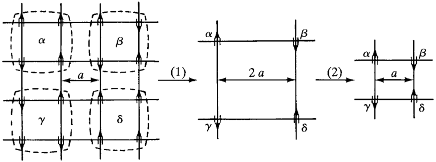
Renormalization group concept

To deal with many degrees of freedom (dof) correlated over different scales:

Renormalization Group (RG).

Integrate iteratively dofs from short to long distances.

E.g. spin block RG.



- The running Hamiltonian contains all couplings allowed by symmetry.
- Over iterations couplings either grow or decay: relevant vs. irrelevant.
- Critical point ≡ fixed point (FP) of the RG transform.

Methods beyond perturbation theory needed to treat non-gaussian FPs.

For $D < 4$ Ising phase transition controlled by strongly-coupled Wilson-Fischer FP.

Èffective action formalism

FRG: implementation of Wilsonian RG

- in momentum space;
- on the effective action $\Gamma[\phi \equiv \langle\varphi\rangle]$.

$$\mathcal{Z}[J] = \int \mathcal{D}[\varphi] \exp\left(-S[\varphi] + \int_x J\varphi\right), \qquad \Gamma[\phi] = -\ln \mathcal{Z}[J] + \int_x J\phi.$$

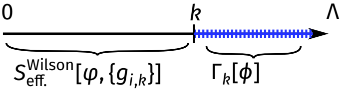
J: external source.

Vertices $\Gamma^{(n)} \equiv \delta^n\Gamma/\delta\phi^n$ contain the physical information:

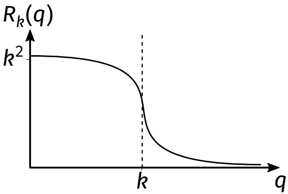
- $\Gamma_{\phi(x)=\mathrm{const.}} = U$: effective potential → thermodynamics.
  Physical $\langle\varphi\rangle$ given by ϕ such that $\partial\Gamma/\partial\phi = 0$.
- $\Gamma^{(2)} = [G]^{-1}$: inverse propagator.

The functional renormalization group: in practice

Similar in concept to Wilsonian RG, degrees of freedom are progressively integrated out.



This is implemented by adding to the action a “mass-like” term:



$$S \to S_k = S + \Delta S_k,$$

$$\Delta S_k[\varphi] = \frac{1}{2}\int_q \varphi(q) R_k(q) \varphi(q).$$

R_k: modes at momenta $\lesssim k$ get a very large mass.

New k-dependent effective action $\Gamma \to \Gamma_k$.

$$\boxed{\Gamma_{k=\Lambda} = S \xrightarrow{\text{RG flow}} \Gamma_{k=0} = \Gamma.}$$

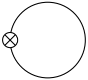
Exact flow equation (Wetterich)

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left\{\partial_t R_k \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}\right\}$$

$t = \ln(k/\Lambda)$: RG “time”.

Flow equation: properties

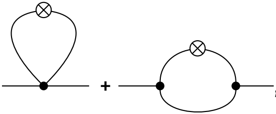
$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\left\{ \partial_t R_k \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right\}$$

- Diagrammatic representation: $\partial_t \Gamma_k = \frac{1}{2}$  → 1-loop structure!

  Line: full propagator G_k; cross: $\partial_t R_k(q)$.

  Replacing $\Gamma_k \to S$ in rhs reproduces 1-loop RG
  → any approx. is at least as good.

- Vertex flow: functional derivatives (≡ inserting leg) then set $\phi \to$ const.

  $\partial_t \Gamma_k^{(2)} = -\frac{1}{2}$  , n-leg dot: vertex $\Gamma^{(n)}$.

  Flow of $\Gamma_k^{(n)} \leftrightarrow \Gamma_k^{(n+1)}, \Gamma_k^{(n+2)}$: hierarchy of equations.

Flow equation: approximations

Simplest approximation schemes:

- Vertex truncation, e.g. $\Gamma_k^{(6)} \to 0$ (or more involved).
  $\to$ simplify field dependence of Γ_k, keep momentum dependence of $\Gamma^{(2)}$.

- For statistical physics, most important:
  Long distance $q \to 0$, field dependence of $U_{k=0}(\phi) \equiv \Gamma[\phi = \text{const.}]$.

  Derivative expansion: expand Γ_k about $p = 0$.

  Justified even close to phase transition: Γ_k is regularized in the IR by R_k.

Derivative expansion

DE: all possible terms to $O(p^{2n})$ allowed by symmetry (here: $\mathbb{Z}_2$).

$$O(p^2):\quad \Gamma_k[\phi] = \int_x \frac{Z_k(\rho)}{2}(\boldsymbol{\nabla}\phi)^2 + U_k(\rho), \quad \rho = \phi^2/2.$$

Additional approx.: no field renormalization term $Z_k(\rho) \to 1$.

Local potential approximation

Ansatz: $\Gamma_k[\phi] = \int_x \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + U_k(\rho), \quad \rho = \phi^2/2.$

k breaks down scale invariance → dimensionless variables $\tilde{q} = q/k$, $\tilde{U} = k^{-D}U$, ...
Flow for $U_k(\rho)$ obtained by projection $q \to 0$:

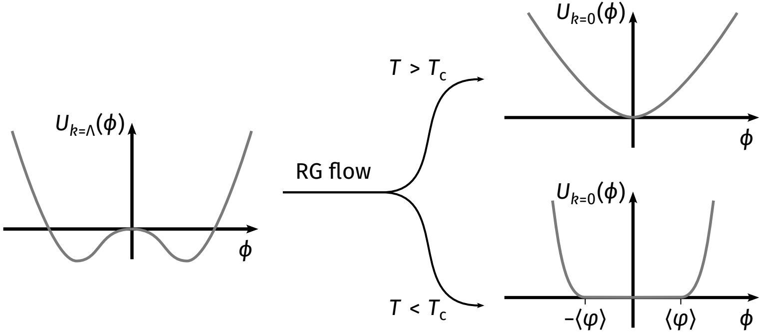
$$\partial_t \tilde{U}_k(\tilde{\rho}) = -2\tilde{U}'_k + (D-2)\tilde{\rho}\tilde{U}'_k + A_D \frac{1}{1+\tilde{U}'_k+2\tilde{\rho}\tilde{U}''_k}$$

$R_k(q^2)=(k^2-q^2)\Theta(k^2-q^2)$,
$A_D=S_{D-1}/D(2\pi)^D$: const.
D:dimension.

Flow of the potential

Effect of the RG flow on a initial condition $U_{k=\Lambda}(\phi) = r_0\phi^2 + u_0\phi^4$:

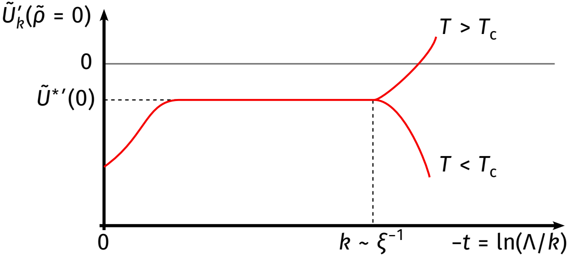
$r_0 = r_{0c} + A(T - T_c)$.



- For $r_0 > r_{0c}$: symmetric phase $\langle\varphi\rangle = 0$.
- For $r_0 < r_{0c}$: broken symmetry phase $\langle\varphi\rangle \neq 0$.
  $U_{k=0}(\phi)$ is convex and minimal at $\pm\langle\varphi\rangle$: “flat bottom” shape.

Fixed point behavior

Phase transition: FP of the flow equations in dimensionless variables.



Linearization of the flow close to FP → critical exponents.

E.g. for $k \to 0$, $\tilde{U}'_k(\tilde{\rho} = 0) = \tilde{U}'_k(0)^* + C(T - T_c)k^{-1/\nu}$.

Is the derivative expansion nonperturbative?

LPA allows for infinite order vertices: $\Gamma^{(n\geq 3)}(p_i,\phi) = \delta_{\sum_i p_i=0}\partial_\phi^n U(\phi)$.

> The nonperturbative character comes from the field dependence of the functions.

$$\partial_t \tilde{U}_k(\tilde{\rho}) = -2\tilde{U}_k' + (d-2)\tilde{\rho}\tilde{U}_k' + A_d \frac{1}{1+\tilde{U}_k'+2\tilde{\rho}\tilde{U}_k''}.$$

$R_k(q^2)=(k^2-q^2)\Theta(k^2-q^2)$.

- Expanding about $\tilde{\rho}=0$, $\tilde{U}_k(\tilde{\rho}) = U_{0,k} + g_{2,k}\tilde{\rho} + g_{4,k}\tilde{\rho}^2/2$:

$$\partial_t g_{4,k} = (D-4)g_{4,k} + \frac{18A_D g_{4,k}^2}{(1+g_{2,k})^4} \longrightarrow \text{polynomial in } g_{4,k} \equiv \text{perturbation theory!}$$

- Expanding about $\tilde{\rho}=\tilde{\rho}_{0,k}$ that minimizes $\tilde{U}_k$ ($\rightarrow$ field dependence):

$$\tilde{U}_k(\tilde{\rho}) = U_{0,k} + \lambda_k(\tilde{\rho}-\tilde{\rho}_{0,k})^2/2 \longrightarrow \text{nonperturbative!}$$

$$\partial_t\tilde{\rho}_{0,k} = -(D-2)\tilde{\rho}_{0,k} + \frac{3A_D}{(1+2\tilde{\rho}_{0,k}\lambda_k)^2}, \quad \partial_t\lambda_k = (D-4)\lambda_k + \frac{6A_D\lambda_k^2}{(1+2\tilde{\rho}_{0,k}\lambda_k)^3}.$$

Difference with fermionic systems

So far: classical systems.

Bosons are treated similarly: 1 complex field = 2 scalar fields.

What is different between bosonic/classical and fermionic systems?

- Grassman fields ψ: only expansion about $\psi = 0$ has meaning, no DE.
- Fermi surface: momentum resolution of vertices important.
- Order parameters: composite fields e.g. $\psi^*_\sigma \psi_\sigma \rightarrow$ phase onset given by diverging susceptibility.

Critical exponents

LPA: $G_k(p) \sim 1/p^2$. How to get η, $G_k(p) \sim 1/p^{2-\eta}$ at the FP?
Include $Z_k(\nabla\phi)^2/2$. Then $G_k(p) \sim 1/Z_k p^2$, at the FP $Z_k \sim k^{-\eta}$, $\partial_t \log Z_k = -\eta$.

Critical exponents for the $D = 3$ Ising model:

	FRG		Monte Carlo	ϵ-exp.	Conformal Bootstrap
	LPA	DE $O(p^6)$			
η	0	0.0361(11)	0.03627(10)	0.0362(6)	0.036298(2)
ν	0.650	0.63012(16)	0.63002(10)	0.6292(5)	0.629971(4)

DE$_6$: [Balog et al. PRL '19], MC: [Hasenbusch PRB '10], ϵ-exp. [Kompaniets and Panzer PRD '17], CB: [Kos et al. JHEP '16].

And beyond critical exponents?

Our work: go beyond and compute operator product expansion (OPE) coefficients in O(N) theories.

Motivation: Operator Product Expansion

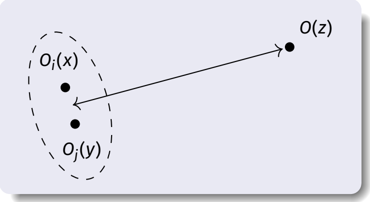
UV divergences → product of operators singular at short distance:

e.g. for a scalar field ϕ: $\lim_{y\to x}\langle\varphi(x)\varphi(y)\rangle = \infty \longrightarrow \varphi(x)\varphi(y)|_{y\to x} \neq \varphi(x)^2$.

Operator Product Expansion (OPE):

for $y \to x$, $$O_i(x)O_j(y) = \sum_k \underbrace{f^{ijk}(x-y)}_{c\text{ number}} \overbrace{O_k(x)}^{\text{local operators}}$$

[Wilson '69, Kadanoff '70]



- sum over all local operators O_k;
- singularities included in f^{ijk}: Wilson coefficients;
- valid when inserted in correlation functions.

Verified to all orders in perturbation theory and in conformal field theories.

OPE in conformal field theories

In conformal field theories (CFT), OPE is the basis for conformal bootstrap (CB).

[Poland et al., RMP '19]

CFT: invariant under transforms that preserve angles. Then:

$$\langle O_i(x) O_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}},$$

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{c_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k} x_{23}^{\Delta_j+\Delta_k-\Delta_i} x_{13}^{\Delta_i+\Delta_k-\Delta_j}}.$$

- Δ_i: scaling dimension.
- $x_{12} = |x_1 - x_2|$.
- c_{ijk}: OPE coefficient!

OPE in CFTs

$$O_i(x) O_j(y) = \sum_k \frac{c_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} O_k(x) + \text{spinful fields}.$$

NB: true even for $d > 2$.

[Di Francesco, Mathieu, Sénéchal, CFT, Springer]

The O(N) model

Similar to φ^4 theory. $\boldsymbol{\varphi}$: N-component real field.

$$S[\boldsymbol{\varphi}] = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left(\partial_\mu \boldsymbol{\varphi}\right)^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

Phase transition controlled by the Wilson-Fisher fixed point for $d < 4$.

- $N = 1, 2, 3$: universality classes of physical systems (Ising, XY, Heisenberg).
- $N = \infty$: exact results.

At the phase transition: emergent conformal invariance!

Most relevant operators: $O_1 \propto \varphi_i$, $O_2 \propto \boldsymbol{\varphi}^2$. $\Delta_1 = (d - 2 + \eta)/2$, $\Delta_2 = d - 1/\nu$.

Question

Coefficient $c_{112} = ?$

- $4 - \epsilon$ expansion; [Dey et al., JHEP '17; Carmi et al., SciPost '21]
- Monte-Carlo; [Caselle et al. PRD '15; Hasenbusch, PRB '20]
- Conformal Bootstrap; [Kos et al. JHEP '16; Cappeli et al., JHEP '19]
- FRG. (Us !)

c_{112} coefficient with FRG

c_{112} can be deduced from correlation functions: [Pagani and Sonoda, PRD '20]

$$\text{for } |p_1| \gg |p_2|, \langle O_1(p_1) O_1(p_2) O_2(-p_1 - p_2)\rangle = \frac{c_{112}\times\text{const.}}{|p_1|^{d-\Delta_2}|p_2|^{d-2\Delta_1}}$$

Strategy: composite operators → add source h. [Rose, Léonard and Dupuis, PRB '15]

$$\mathcal{Z}[J,h] = \int \mathcal{D}[\varphi]\, e^{-S[\varphi]+\int_x (J\varphi + h\varphi^2)} \to \text{Legendre transf.: } \Gamma[\phi, h].$$

$$\langle \varphi_i(p_1)\varphi_i(p_2)\varphi^2(-p_1-p_2)\rangle = -\overbrace{G(p_1)}^{G:\ \text{propagator}}\ \underbrace{\Gamma^{(2,1)}_{ii}(p_1,p_2)}_{\Gamma^{(2,1)}_{ii} = \delta^3\Gamma/\delta\phi_i\delta\phi_i\delta h|_{\boldsymbol{\phi}=\text{const}, h=0}:\ \text{vertex}}\ G(p_2)$$

Setting $p_2 = 0$, $p_1 = p \to 0$:

$$\boxed{c_{112} = \text{const.} \times \lim_{p\to 0} \frac{\Gamma^{(2,1)}_{ii}(p,0)}{|p|^{\Delta_2 - 2\Delta_1}}}$$

Momentum dependence → BMW scheme [Blaizot, Méndez-Galain and Wschebor, PLB '06]

- First determine $G(p)$ and $\chi_s = \langle \boldsymbol{\varphi}^2 \boldsymbol{\varphi}^2\rangle$ → normalization of operators, Δ_i.
- Then $\Gamma^{(2,1)}_{ii}(p,0) = \partial_{\phi_i}\Gamma^{(1,1)}_i(p) \to c_{112}$.

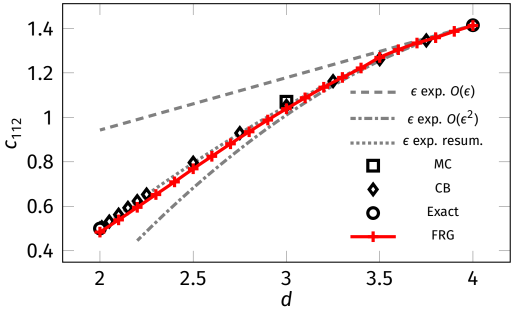
Ising model: universality class of $N = 1$.

Known values:

- $d = 2$, exact solution;
- $d = 4$, mean-field.

$2 < d < 4$:

- Monte-Carlo, Conformal bootstrap: numerically exact, but expensive;
- $\epsilon = 4 - d$ expansion: requires resummation and $d = 2$ result.

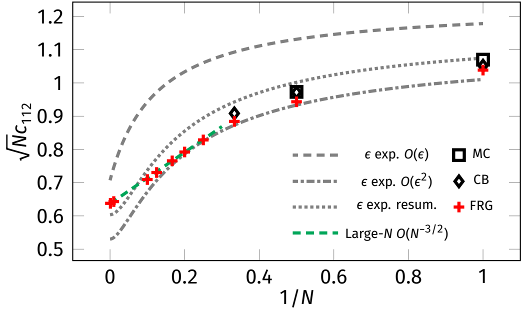


Results: c_{112} in the $3d$ O(N) model vs. N

Large N: $$c_{112} = \frac{2}{\pi}\frac{1}{\sqrt{N}} + \frac{24}{\pi^3}\frac{1}{N^{3/2}} + O\left(\frac{1}{N^{5/2}}\right)$$

[Lang and Rühl, Nucl. Phys. B ’92]

→ rescaling: $\sqrt{N}c_{112}$.



- $N = 1, 2, 3$: agreement with CB and MC.
- $N \gtrsim 10$: agreement with large N results to next-to-leading order.
- Failure of ϵ expansion.

Other classical statistical physics applications

Beyond O(N) models?

- Disordered systems, e.g. random field $h(x)$ coupled to the order parameter.
  Important: cumulants of $\ln Z[J; h] \to$ replica trick.
  Random-field Ising model: $\to$ breakdown of dimensional reduction in $D = 3$.
  [Tissier, Tarjus PRB '08, PRL '11, ...]

- Langevin stochastic dynamics: Martin-Siggia-Rose-De Dominicis-Janssen.

  Langevin equation $\to$ field theory.

  E.g. turbulence in driven Navier-Stokes.
  - Exact symmetry-base Ward identities between correlation functions.
  - Time dependence of correlation functions. [Canet et al., PRE '16]

Quantum criticality

FRG for quantum criticality?

- Quantum phase transition (QPT): qualitative change of ground state as an external nonthermal parameter is tuned.
- Trotterization: d-dimensional quantum problem ≡ $d + 1$ classical field theory.
- Price: new imaginary time dimension $\tau \in [0, \beta]$.
- QPT: a phase transition in the classical field theory.

→ QPTs show all features of classical critical phenomenons:

universality classes, scaling,...

Quantum O(N) model

Lorentz-invariant action, where $\boldsymbol{\varphi}$ is a real N-component field ($\sim \phi^4$ model)

$$S[\boldsymbol{\varphi}] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{r} \left\{ \frac{1}{2} (\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{1}{2c^2} \left(\partial_\tau \boldsymbol{\varphi}\right)^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

- Temperature-independent couplings.
- Effective action that describes several condensed matter phase transitions:
  Bose–Hubbard model ($N = 2$), Quantum antiferromagnets ($N = 3$).

Experimental realizations:

- $N = 2$: cold atoms [Endres et al. Nature '12], 2d GaAs [Lagoin et al., Nature Phys. '22].
- $N = 3$: quantum antiferromagnets [Rüegg et al. PRL '08, Hong et al. Nat. Phys. '17].

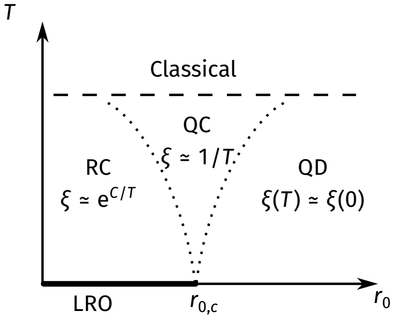
$T = 0$ model equivalent to the $d + 1$ classical O(N) model.

Phase transition: $\sim$ Mott insulator $\langle \boldsymbol{\varphi} \rangle = 0$ to superfluid $\langle \boldsymbol{\varphi} \rangle \neq 0$.

Qualitative $T > 0$ phase diagram

At finite T the QPT is destroyed (critical energy scales $\ll T$).

However there are crossovers between different regimes:



Typical ($N \geq 3$) $2d$ phase diagram.

- Energy scales: T and zero-T gap Δ.
- Crossover lines $T \sim \Delta \sim |\delta r_0|^\nu$.
- For $N = 1$ or 2 existence of a $T > 0$ "ordered" phase.

Near the QPT:

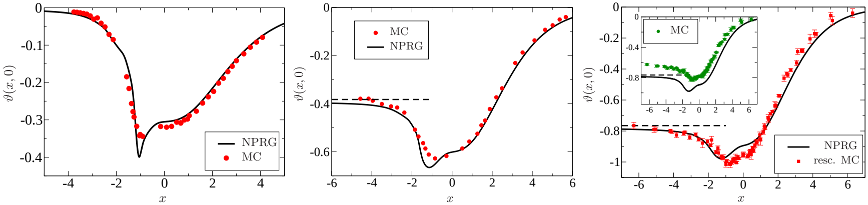
$$P(T) = P(T = 0) + T^{2+z}\mathcal{F}\left(\frac{T}{\Delta}\right), \qquad \sigma(\omega) = \frac{q^2}{h}\Sigma\left(\frac{\omega}{\Delta}, \frac{T}{\Delta}\right)$$

with $\mathcal{F}(x)$, $\Sigma(x, y)$ universal scaling functions.

Application of the DE: universal scaling functions of the pressure and internal energy density.

$$P(T) = P(T=0) + \frac{(k_B T)^3}{(\hbar c)^2}\mathcal{F}(\Delta/(k_B T), \qquad \epsilon(T) = \epsilon(T=0) - \frac{(k_B T)^3}{(\hbar c)^2}\vartheta(\Delta/k_B T).$$

Left to right: $\vartheta(x)$ for Ising, XY, Heisenberg $[x = (\Delta/k_B T)^{1/\nu}]$.



[Rançon, ..., Rose et al., PRB '16].

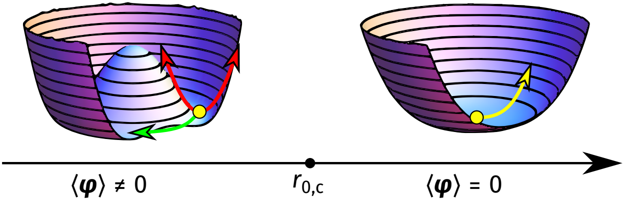
Full lines: FRG; dots: Monte-Carlo simulations for $3D$ classical spin systems with PBC.

DE $O(\partial^2)$, improvement over previous work by Rançon et al.

Dynamics: mean-field excitations

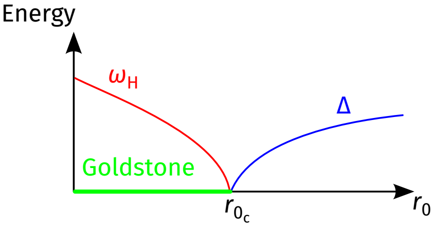
$$U(\boldsymbol{\phi}) = r_0 \boldsymbol{\phi}^2 + u_0 (\boldsymbol{\phi}^2)^2$$

Mean-field zero T phase diagram:



$N - 1$ Goldstone modes
+ amplitude mode (gap ω_H).

N gapped modes (gap Δ).



Mean-field:

$$\Delta = A \left| r_0 - r_{0,c} \right|^{1/2}$$

$$\omega_H = A\sqrt{2} \left| r_0 - r_{0,c} \right|^{1/2}$$

Beyond the Gaussian approximation

Question: what happens to the "Higgs" amplitude mode beyond MF?

- Is it a well defined mode? What is the quasiexcitation lifetime?
- What happens near the critical point?

> "In general, this Higgs particle can decay into multiple lower-energy spin waves. It has been argued that such decay processes dominate for $d < 3$, and the Higgs particle is therefore not a stable excitation."

[Sachdev, *Quantum Phase Transitions*, 2nd. ed]

Emission of Goldstone bosons → IR divergence of the longitudinal susceptibility.

[Patasinskij *et al.*, JETP' 73], [Zwerger, PRL '04], [Dupuis, PRE '11], ...

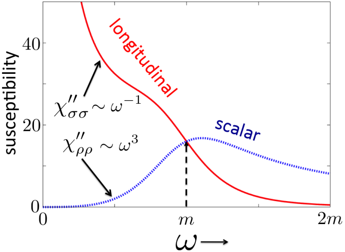
Scalar response function

Answer: consider a different response function. [Podolsky, Auerbach and Arovas, PRB '11]

Right probe: scalar susceptibility.

$$\chi_s(\mathbf{r},\tau) = \langle \boldsymbol{\phi}^2(\mathbf{r},\tau)\boldsymbol{\phi}^2(0,0)\rangle,$$

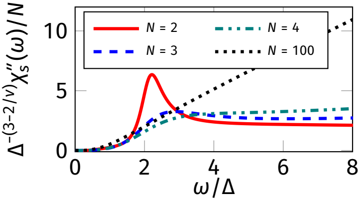
$$\chi_s''(\omega) = \mathrm{Im}[\chi_s(\mathbf{q}=0, i\omega_n \to \omega + i0^+)].$$



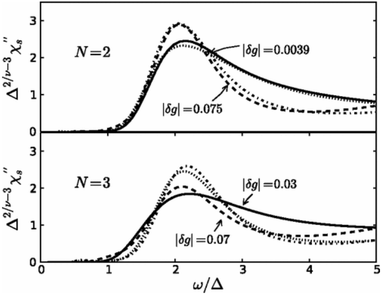
- Disagreement between large-N and weak coupling.
- Determination of χ_s: source term $\int_x h\varphi^2$, analog to OPE calculation.
- Analytic continuation: Padé approximants ($T=0$).

Results and comparison

FPRG BMW [Rose and Dupuis, PRB ’15]:



MC [Gazit, Podolski, Auerbach PRL ’13]:



Experimental observation: [Endres et al., Nature ’12].

Higgs mass m_H/Δ	$N=3$	$N=2$		$N=3$	$N=2$
MF	$\sqrt{2}$	$\sqrt{2}$	FRG BMW	2.7	2.2
QMC (Chen et al.)		3.3(8)	Lattice QMC (Löhofer et al.)	2.6(4)	
QMC (Gazit et al.)	2.2(3)	2.1(3)	Exact diag. (Nishiyama)	2.7	2.1(2)
ϵ-exp. (Katan et al.)	1.64	1.67			

Review and conclusion

- FRG: powerful tool → nonperturbative approximations.
- Wide success for critical phenomena: from critical exponents to OPEs.

  [Rose, Pagani and Dupuis PRD '22]
- Applicable to a large class of systems.
- Successes in quantum bosonic systems: thermodynamics [Rançon, ..., Rose et al. PRB '16] and dynamics [Rose Léonard and Dupuis PRB '15, Rose and Dupuis PRB '17].
- Long term goal: $T > 0$ transport.

  In the quantum O(N) model, quantum critical regime: Planckian transport.

  Big issue: analytic continuation! Proposals to overcome this difficulty (Strodthoff, Pawlowski).

Thanks for your attention!

Applications of OPE

Renormalization theory [Brandt, Ann Phys '67], chromodynamics [Novikov et al., PR '78].

Ultracold gases: thermodynamic relations for 3d interacting fermions.

[Braaten and Platter, PRL '08]

$$\text{OPE: } \psi_\sigma^\dagger(\mathbf{R} - \tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) = \textstyle\sum_i C_i(\mathbf{r}) O_i(\mathbf{R}).$$

Operator identity: $C_i(\mathbf{r})$ determined by evaluating with few-body scattering states.

Result:

$$\psi_\sigma^\dagger(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r}) = \psi_\sigma^\dagger\psi_\sigma(\mathbf{R})$$
$$+\, \mathbf{r} \cdot [\psi_\sigma^\dagger \overleftrightarrow{\boldsymbol{\nabla}} \psi_\sigma](\mathbf{R}) - \frac{r}{8\pi} g^2 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\uparrow \psi_\downarrow(\mathbf{R}) + O(r^2).$$

g: contact interaction.

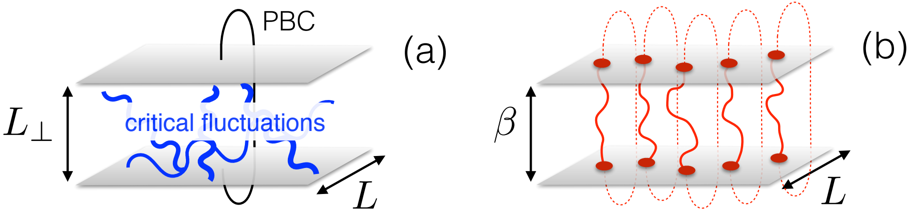
E.g.: high-frequency tail of momentum distribution, $\rho_\sigma(\mathbf{k}) \sim C/k^4$.

$C = \int_\mathbf{R} \langle g^2 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\uparrow \psi_\downarrow \rangle$: Tan contact, $\quad \partial_a \langle \hat{H} \rangle = (\hbar^2/4\pi m a^2) C.$ [Tan, Ann Phys '08]

Classical — quantum mapping

Classical theory confined along a $L_\perp$ direction equivalent to quantum $T > 0$ theory!

$$H_{\mathrm{cl.}} = \int \mathrm{d}^{D-1}\mathbf{x}_\parallel \int_0^{L_\perp} \mathrm{d}x_\perp \, \mathcal{H}, \qquad H_{\mathrm{q.}} = \int \mathrm{d}^d\mathbf{r} \int_0^{\hbar\beta} \mathrm{d}\tau \, \mathcal{H}.$$



The scaling function ϑ describes the scaling of the critical Casimir force of a $3D$ classical model near criticality with periodic boundary conditions.

$$\text{Casimir force } f(L_\perp, \xi) \sim L_\perp^{-D} \vartheta(L_\perp/\xi).$$

(Critical Casimir forces: [Fisher and de Gennes, C.R. Acad. Sci. '78]) ξ: correlation length.