Probability distribution function of the 2*d* Ising order parameter

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Recap from Adam's talk

Sums of random variables

$(X_1 + \cdots + X_n) \to ?$

- Universality: small number of limit distributions.

- Beyond CLT/Lévy distributions: strongly correlated variables
  →e.g Ising spins!

$$\hat{s}_i = \pm 1, \qquad P(\{\hat{s}_i\}) \propto e^{-\beta H(\{\hat{s}_i\})}, \qquad H = -J \sum_{\langle ij \rangle} \hat{s}_i \hat{s}_j$$

L: box size,
d: dimension.

$$\hat{s} = \frac{1}{L^d} \sum_i \hat{s}_i,$$

$P(\hat{s} = s) = ?$

Rate function

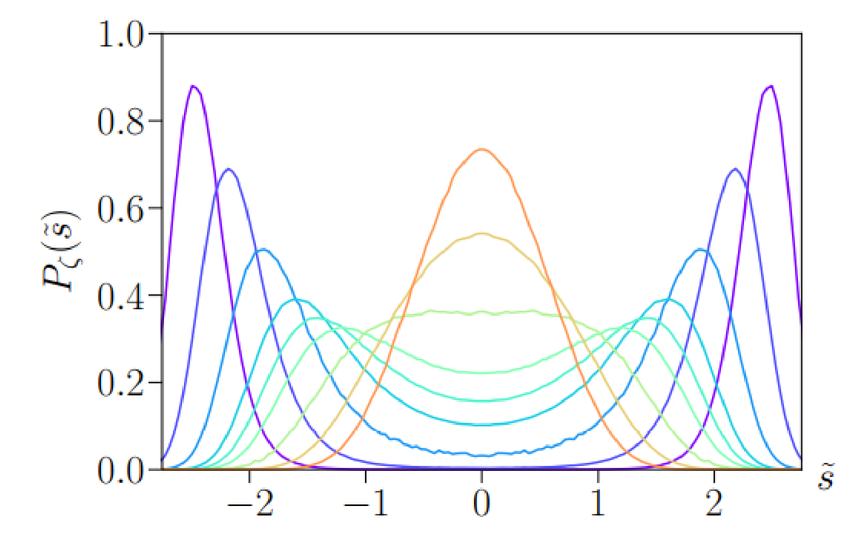
Critical Ising spins, correlation length ξ_∞.

Rate function: $P_\zeta(\hat{s} = s) \simeq \exp\left(-L^d I(s, \xi_\infty, L)\right)$.

Scaling hypothesis: $L^d I(s, \xi_\infty, L) = I_\zeta(\tilde{s})$.

$\tilde{s} = L^{(d-2+\eta)/2} s$, $|\zeta| = L/\xi_\infty$.

sign(ζ) = (–)1 in the (broken) symmetry phase.



$P_\zeta(\tilde{s})$ vs $\tilde{s}$ for ζ=-4,-3,...4.

3*d* Ising model, periodic boundaries.

Monte-Carlo simulations.

[Balog, Rançon, Delamotte, PRL '22]

FRG approach for the rate function

- S: ϕ^4 theory, describes Ising near criticality.

Order parameter: magnetization $\langle\hat{\phi}\rangle$.

Sum of spins $\to$ $\hat{s} = L^{-d} \int_x \hat{\phi}(x)$.

$$P(\hat{s} = s) = \mathcal{N} \int \mathcal{D}[\hat{\phi}]\delta(s - \hat{s})e^{-S[\hat{\phi}]} = \lim_{M\to\infty} \mathcal{N} \int \mathcal{D}[\hat{\phi}]e^{-\frac{M^2}{2}(s-\hat{s})^2} e^{-S[\hat{\phi}]}.$$

FRG approach for the rate function

$$S_M[\hat{\phi}] = S[\hat{\phi}] + \frac{M^2}{2}\left(\int_x (\hat{\phi}(x) - s)\right)^2$$

- $M = 0$: original action.
- $M \to \infty$: PDF, zero mode frozen!

- FRG: regulator R_k, RG scale k, modified Legendre transform

$$\Gamma_{M,k}[\phi] = -\ln \mathcal{Z}_{M,k}[J] + \int_x J_x \phi_x - \frac{1}{2}\sum_q \phi_q \phi_{-q} R_k(q) - \frac{M^2}{2}\left(\int_x (\phi_x - s)\right)^2 .$$

- $\Gamma_{M,k}$ defined to have good limits for M, k large.
- $\Gamma_{M,k}$ is independent of s!

Constraint effective action

$\check{\Gamma}_k = \lim_{M\to\infty} \Gamma_{M,k}$: constraint effective action.

- Constraint: $M \to \infty$ ~ large mass only for the mode $q = 0$.
- Flow equations are the same with zero mode frozen
  → explicit box size dependency!

$$\check{\Gamma}(\phi \to \text{const.}) = L^d I_\zeta(\phi).$$

Flow of the rate function: LPA

$$\partial_k \check{\Gamma}_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left\{\partial_k R_k \left(\check{\Gamma}_k^{(2)}[\phi] + R_k\right)^{-1}\right\}$$

Propagator $G_k(q) = (\Gamma_k^{(2)}(q) + R_k(q))^{-1}$.

Within LPA $\Gamma_k^{(2)}(q) = q^2 + \partial_\phi^2 U_k$.

$\Gamma_k \to \check{\Gamma}_k$: replace $U_k \to I_k$.

[see also Fister and Pawlowski '15]

$$\partial_k U_k[\phi] = \frac{1}{2}\sum_q \partial_k R_k(q) G_k(q) \qquad \partial_k I_k[\phi] = \frac{1}{2}\sum_{q\neq 0} \partial_k R_k(q) G_k(q)$$

- Equations identical up to the removal of the zero mode.
- L^{-1} acts as an infrared cutoff: $I_{k\to 0}$ has a finite limit.

$q = \frac{2\pi}{L}(n_1, \dots, n_d)$,
$n_i \in \mathbb{Z}$.

$d = 3$:
LPA works well!

Going to $d = 2$

$d = 2$?

- $Z[h] = \langle e^{hs} \rangle$: moment generating functional of $P(s)$.
- Interest : stronger correlations; no exact results.
- LPA not enough! Need to include field corrections: $\eta^{\mathrm{LPA}} = 0$!

How to deal with discrete modes

- Beyond LPA: derivative expansion.

- Gradient expansion of Γ_k: discrete modes ?

$$\Gamma_k^{\mathrm{DE}_2}[\phi] = \int_x \frac{Z_k(\phi)}{2}(\nabla\phi)^2 + U_k(\phi).$$

Question:
What does it mean to expand at small q when q is discrete?

A concrete case: DE_2

$$\check{\Gamma}_k^{\mathrm{DE}_2}[\phi] = \int_x \frac{Z_k(\phi)}{2}(\nabla\phi)^2 + I_k(\phi)?$$

$Z_k(\phi) = ?$

- Discrete variable: $Z_k(\phi) = \partial_{p^2}\check{\Gamma}_k^{(2)}(p;\phi)|_{p\to 0}$ not allowed.

$$p_n = \frac{2\pi}{L}(n, 0, \dots, 0).$$

$$Z_k(\phi) = \frac{\check{\Gamma}_k^{(2)}(p_1;\phi) - \check{\Gamma}_k^{(2)}(0;\phi)}{p_1^2}?$$

Propagator flow equation

- No! Due to zero mode discrepancy between $p = 0$ and $p \neq 0$!

$$\partial_k \check{\Gamma}_k^{(2)}(p;\phi) = \frac{1}{2L^d} \sum_{q\neq 0} \partial_k \check{G}_k(q;\phi) \check{\Gamma}_k^{(4)}(p,-p,q,-q;\phi)$$

$$- \frac{1}{2L^d} \sum_{q\neq 0,-p} \partial_k \left(\check{G}_k(q;\phi)\check{G}_k(q+p;\phi)\right) \check{\Gamma}_k^{(3)}(p,q,-q-p;\phi)\check{\Gamma}_k^{(3)}(-p,-q,p+q;\phi).$$

- Formally, for $p > 0$

$$\check{\Gamma}_k^{(2)}(p;\phi) - \check{\Gamma}_k^{(2)}(0;\phi) \simeq \Delta_{0,k}(\phi) + p^2 Z_k(\phi) + O(p^4).$$

(Recall $\check{\Gamma}_k^{(2)}(0;\phi) = I_k''(\phi)$.)

DE$_2$ parameterization

- Solution: *Ansatz*,

$$\check{\Gamma}_k^{(2)}(p;\phi) = \begin{cases} I_k''(\phi) & \text{if } p = 0, \\ I_k''(\phi) + \Delta_{0,k}(\phi) + Z_k(\phi)p^2 & \text{otherwise.} \end{cases}$$

- Flows of $\Delta_{0,k}(\phi)$, $Z_k(\phi)$ deduced from $\check{\Gamma}_k^{(2)}(p_n;\phi)$ for $n = 0, 1, 2$.

- Differs from "actual" DE$_2$ *Ansatz*: vertices have to be inferred!

e.g. $\check{\Gamma}_k^{(3)}(p, q, -p-q;\phi) = I_k'''(\phi) + \Delta_{0,k}'(\phi) + Z_k'(\phi)(p^2 + q^2 + p\cdot q)$.

BMW approach

- Other idea: the celebrated Blaizot-Méndez-Galain-Wschebor (BMW) approximation. [Blaizot et coll., PRE ’06] [Benitez et coll., PRE ’09]

- Close flow equations of $\check{\Gamma}_k^{(2)}(p_n)$: (Not a vertex truncation!)
  full momentum dependence!

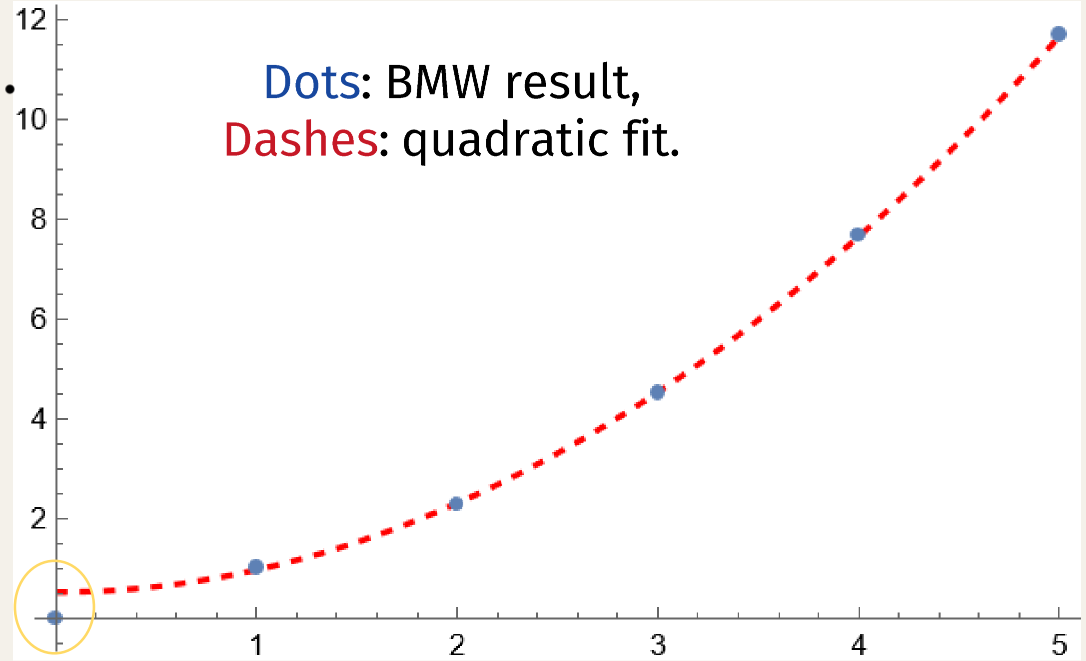
- Solves by “brute force” the zero-mode problem.

BMW results

- Blaizot-Méndez-Galain-Wschebor (BMW) approximation:

  Full momentum dependence of $\Gamma_k^{(2)}(p_n)$.

$$\check{\Gamma}^{(2)}_{k=0}(p_n;\phi) - \check{\Gamma}^{(2)}_{k=0}(0;\phi) \text{ vs. } n.$$



- Discrepancy between $n = 0$ and $n > 0$!

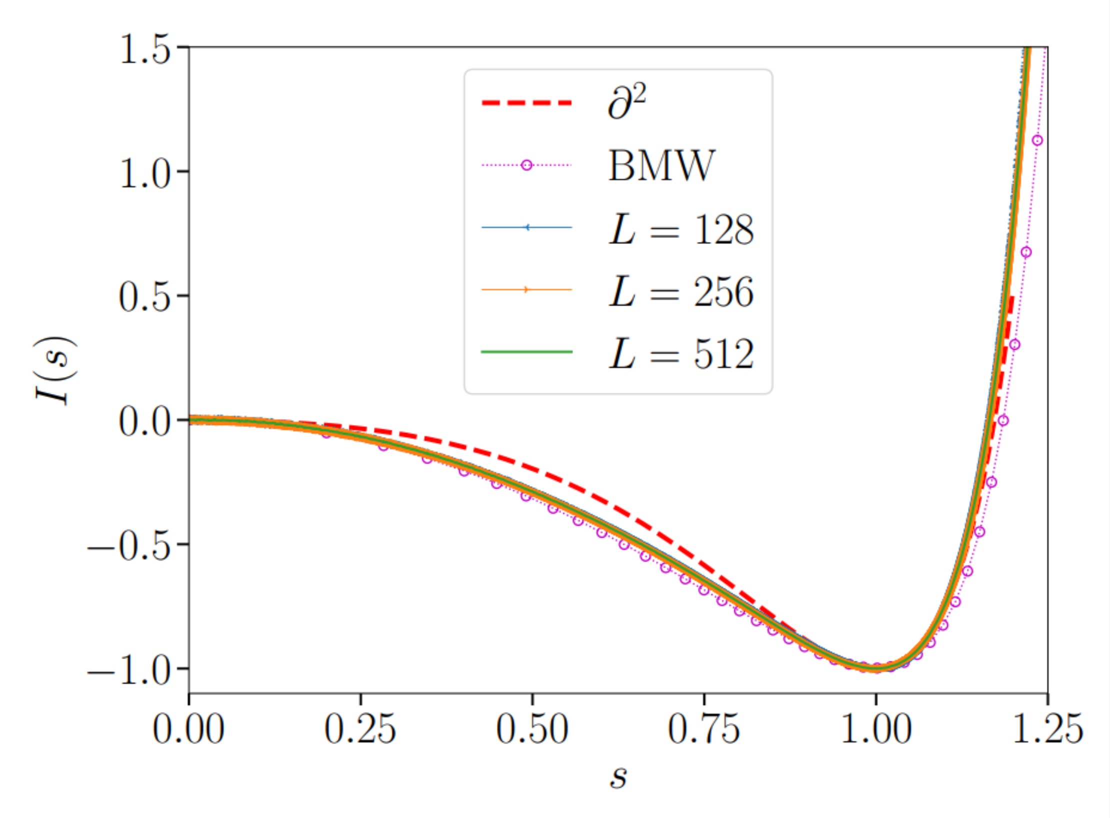
$$p_n = \frac{2\pi}{L}(n, 0, \dots, 0).$$

BMW preliminary results

- Rate function:
  DE$_2$ vs. BMW vs. MC.

BMW: sensible improvement over DE$_{2}$!

Preliminary results.



$I_{\zeta=0}(\tilde{s})$ vs. $\tilde{s}$.

Similarity to QFTs at T>0

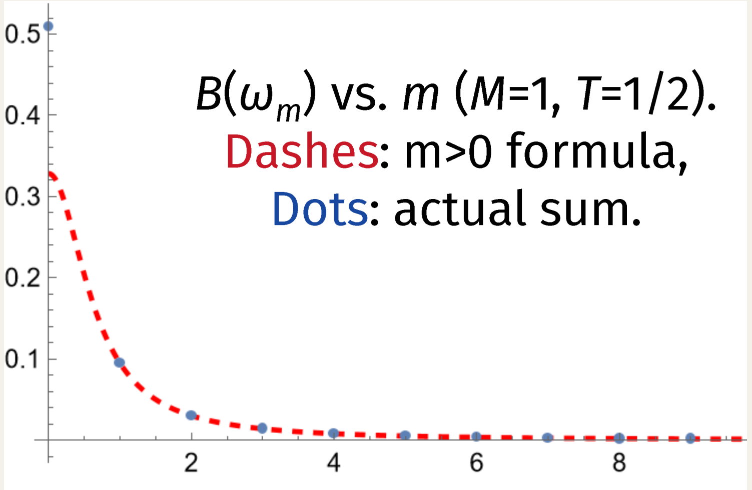
- Inverse temperature = box size.

$$G(i\omega_n) = \frac{1}{\omega_n^2 + M^2}$$

$$i\omega_n = 2\pi n T$$

$$B(\omega_m) = T \sum_n G(i\omega_n) G(i\omega_n + i\omega_m)$$

$$= \begin{cases} \dfrac{\coth(M/2T)}{4M^3} + \dfrac{1}{8TM^2 \sinh^2(M/2T)} & \text{if } m = 0, \\ \dfrac{\coth(M/2T)}{M(4M^2 + \omega_m^2)} & \text{if } m \neq 0. \end{cases}$$



$B(\omega_m)$ vs. m (M=1, T=1/2).
Dashes: m>0 formula,
Dots: actual sum.

- Response functions: static vs. dynamic $\omega \to 0$ responses.

[Dupuis, Field Th. Of Cond. Mat. '23]

Conclusion

- Finite size matters: needs momentum dependency.
- Success of BMW!
- Connection to QFT at $T > 0$: discrete Matsubara frequencies, derivative expansion justified?

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Manuscript in preparation!

Thank you for your attention!